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Diffraction of Kelvin waves from a rotating channel with an infinite and a semi-infinite barrier

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Abstract. The diffraction of long waves in a rotating channel of constant depth, consisting of an infinite and a semi-infinite parallel barrier of zero thickness, is investigated. The excitation is due to a Kelvin wave travelling towards the open end from inside the channel.

An exact solution is given for the region inside the semi-infinite channel, consisting of a 'reflected' Kelvin wave and the various travelling modes, which become attenuated waves when the width of the channel is less than a certain value. Moreover it is shown that there is a resonance effect for some critical values of the channel width. In the remaining region an asymptotic expression for the far field is obtained. This expression involves a Kelvin wave travelling along the infinite barrier and also an outgoing cylindrical wave (Poincaré wave) coming from the edge of the semi-infinite barrier of the channel.

1. Introduction

In this paper the diffraction of long waves in a rotating channel of constant depth, consisting of an infinite and a semi-infinite parallel plane of zero thickness, is investigated. The excitation is due to a Kelvin wave travelling towards the open end from inside the channel.

An exact solution is given for the region inside the semi-infinite channel; in the remaining region an asymptotic expression for the far field is obtained. These theoretical results may explain qualitatively the formulation of tides in the case of an elongated island along a straight coastline.

The method followed is based on the Wiener–Hopf technique, and the appropriate functional equation is derived by applying the complex Fourier transform directly to the differential equation of the problem.

In the particular case of zero angular velocity the solution concerns the surface elevation without rotation and also represents the propagation of electromagnetic or sound waves produced by a plane harmonic wave incident along the above channel, studied by Heins (1956).

2. Formulation of the problem

The linearised equations of motion for long waves in a sheet of water of uniform depth,

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assuming a time factor $e^{-i\omega t}$, are, in Cartesian coordinates (Proudman 1953),

$$hk^2u = -i\omega\, \partial\phi_t/\partial x + f\, \partial\phi_t/\partial y \tag{2.1a}$$

$$hk^2v = -f\, \partial\phi_t/\partial x - i\omega\, \partial\phi_t/\partial y \tag{2.1b}$$

$$(\nabla^2 + k^2)\phi_t = 0\,. \tag{2.1c}$$

Here ϕ_t is the total elevation of the free surface above its mean level; u, v are the components of the velocity in the horizontal x, y plane, which are functions of x, y, t only; f is the Coriolis parameter equal to $2\Omega \sin \rho$, where Ω is the angular velocity of the Earth and ρ the northern latitude; and ∇^2 is the two-dimensional Laplacian. Also $c^2k^2 = \omega^2 - f^2$, $c^2 = gh$ and $\omega > f$, where g is the gravitational acceleration.

Now suppose that a Kelvin wave $\phi_i = \exp[c^{-1}(i\omega x - fy)]$ moves to the right towards the open end of a channel formed by an infinite plane $(y = 0, -\infty < x < \infty)$ and a semi-infinite plane $(y = b,\ x < 0)$ (figure 1). The fluid occupies the field $y \geqslant 0$, and we investigate the waves radiated from this model.

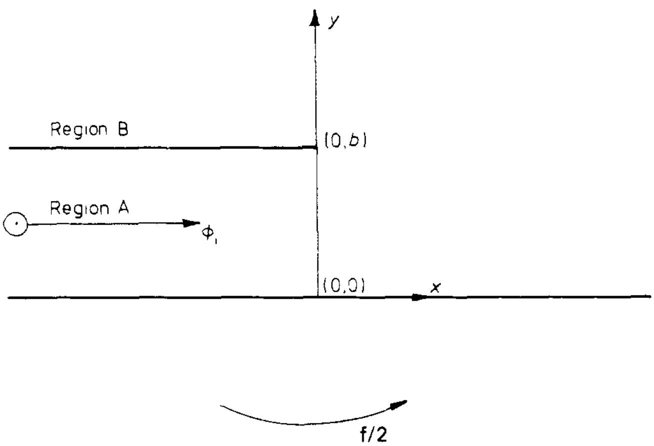

Figure 1. The geometry of the channel.

The whole field is divided into two regions—region A $(0 \leqslant y \leqslant b, -\infty < x < \infty)$ and region B $(y \geqslant b, -\infty < x < \infty)$—and the function ϕ is defined by

$$\phi_t = \begin{cases} \phi + \phi_i \text{ in region A} \\ \phi \text{ in region B.} \end{cases} \tag{2.2}$$

For convenience we put $f = kc \sinh \beta$, $\omega = kc \cosh \beta$; the incident wave is then written as $\phi_i = \exp[k(ix \cosh \beta - y \sinh \beta)]$.

The problem is to find a solution of equation (2.1c) satisfying the following conditions:

$$\frac{\partial\phi_t}{\partial y} - i \tanh \beta \frac{\partial\phi_t}{\partial x} = 0 \qquad \begin{cases} \text{for } y = 0 & (-\infty < x < \infty) \\ \text{for } y = b \pm 0 & (x < 0) \end{cases} \tag{2.3}$$

$$\partial\phi_t/\partial y - i(\tanh \beta)\, \partial\phi_t/\partial x \text{ is continuous across } y = b \qquad (-\infty < x < \infty) \tag{2.4}$$

$$\text{and } \phi_t \text{ is continuous across } y = b \qquad (x \geqslant 0). \tag{2.5}$$

Lastly we assume

$$\phi_t = O(1) \text{ as } x \to 0 \pm 0 \text{ on } y = b \tag{2.6a}$$

and

$$\partial\phi_t/\partial y = O(x^{-1/2}) \text{ as } x \to 0+0 \text{ on } y = b. \tag{2.6b}$$

According to definition (2.2) the above conditions also hold for ϕ.

In the following we shall assume that ω is complex with a small positive imaginary part ω_2 (i.e. $\omega = \omega_1 + i\omega_2$; $\omega_1 \gg \omega_2 > 0$). This implies that k is also complex with a small positive imaginary part k_2 (i.e. $k = k_1 + ik_2$; $k_1 \gg k_2 > 0$). The solution is obtained from the final results by making $\omega_2 \to 0+0$, which implies $k_2 \to 0+0$ and vice versa.

From (2.2) and the result found by the author (Kapoulitsas 1975) that all the modes travelling into a semi-infinite channel from the origin to the left are of order $e^{\tau_0 x}$ as $x \to -\infty$, where $\tau_0 < k_2(1 - f^2/\omega_1^2)^{1/2} < k_2$, it can easily be seen that, for any fixed y, $\phi = O(e^{-\tau_0|x|})$ at most, as $|x| \to \infty$; hence the two-sided complex Fourier transform $\Phi(\alpha, y)$ of $\phi(x, y)$ in x, defined by

$$\Phi(\alpha, y) = \int_{-\infty}^{\infty} \phi(x, y)\, e^{i\alpha x}\, dx, \qquad \alpha = \sigma + i\tau \qquad (\sigma, \tau \text{ real}), \tag{2.7}$$

exists in the whole field and is regular in the strip $|\tau| < \tau_0$ of the α plane. Moreover it is found that ϕ is bounded as $y \to \infty$ for any α in the above strip of regularity.

We also define the one-sided complex Fourier transforms

$$\Phi_+(\alpha, y) = \int_0^{\infty} \phi(x, y)\, e^{i\alpha x}\, dx \tag{2.8a}$$

and

$$\Phi_-(\alpha, y) = \int_{-\infty}^{0} \phi(x, y)\, e^{i\alpha x}\, dx \tag{2.8b}$$

which are regular in $\tau > -\tau_0$ and $\tau < \tau_0$ respectively. Evidently

$$\Phi(\alpha, y) = \Phi_+(\alpha, y) + \Phi_-(\alpha, y). \tag{2.9}$$

3. The basic Wiener–Hopf equation

Applying the two-sided Fourier transform to equation (2.1*c*), and supposing that $\partial\phi/\partial x$ has a similar behaviour to ϕ for $|x| \to \infty$, we get

$$d^2\Phi(\alpha, y)/dy^2 - \gamma^2\Phi(\alpha, y) = 0 \qquad (|\tau| < \tau_0) \tag{3.1}$$

where

$$\gamma = (\alpha^2 - k^2)^{1/2}. \tag{3.2}$$

The function γ has branch points at $\alpha = \pm k$; the cuts from these points are taken symmetrical to each other with respect to the origin, outside the strip $|\tau| < k_2$, and along the straight line passing through these branch points. To have a physically acceptable solution of the problem—that is, the solution which satisfies all the requirements of the previous sections as well as Sommerfeld's 'radiation condition' at infinity—a suitable branch of the primarily multivalued function $(\alpha^2 - k^2)^{1/2}$ is specified by the author (Kapoulitsas 1975).

We mention here that by this specification the real part of γ is positive inside the strip $|\tau| < k_2$.

The general solution of (3.1), regular in $|\tau| < \tau_0$, is of the form

$$\Phi(\alpha, y) = \begin{cases} A(\alpha)\,e^{-\gamma y} + B(\alpha)\,e^{\gamma y} & (0 \leqslant y \leqslant b, \text{ region A}) \\ C(\alpha)\,e^{-\gamma y} & (y \geqslant b, \text{ region B}) \end{cases} \tag{3.3}$$

where A, B and C are functions of α but not of y.

On applying the two-sided Fourier transform to the boundary condition (2.3) we get

$$\Phi'(\alpha, 0) - \alpha(\tanh\beta)\Phi(\alpha, 0) = 0 \tag{3.4}$$

where the prime is taken to mean the derivative with respect to y.

Using now (3.3) we obtain

$$B = A(\gamma + \alpha \tanh\beta)/(\gamma - \alpha \tanh\beta). \tag{3.5}$$

Also from condition (2.4) we have

$$\Phi'(\alpha, b-0) - \alpha(\tanh\beta)\Phi(\alpha, b-0) = \Phi'(\alpha, b+0) - \alpha(\tanh\beta)\Phi(\alpha, b+0), \tag{3.6}$$

and using (3.3) and (3.5) we obtain

$$C = -2A\,e^{\gamma b}(\sinh\gamma b). \tag{3.7}$$

Next from (2.3) applied to $y = b \pm 0$ we have, after taking the one-sided Fourier transform from $-\infty$ to 0,

$$\Phi'_-(\alpha, b\pm 0) - \alpha(\tanh\beta)\Phi_-(\alpha, b\pm 0) - \mathrm{i}(\tanh\beta)\phi(0, b) = 0, \tag{3.8}$$

since $\phi(0, b-0) = \phi(0, b+0) = \phi(0, b)$.

From equations (3.8) and (3.6) we obtain

$$\begin{aligned} &\Phi_+(\alpha, b-0) - \alpha(\tanh\beta)\Phi_+(\alpha, b-0) + \mathrm{i}(\tanh\beta)\phi(0, b) \\ &\quad = \Phi_+(\alpha, b-0) - \alpha(\tanh\beta)\Phi_+(\alpha, b+0) + \mathrm{i}(\tanh\beta)\phi(0, b) \\ &\quad \equiv \Psi_+(\alpha, b), \text{ say.} \end{aligned} \tag{3.9}$$

Finally from condition (2.5) we obtain

$$\Phi_+(\alpha, b-0) + \mathrm{i}\exp(-kb\sinh\beta)/(k\cosh\beta + \alpha) = \Phi_+(\alpha, b+0). \tag{3.10}$$

Now using equations (3.3), (3.8) and (3.9) we get

$$\Psi_+(\alpha, b) = 2A(\gamma + \alpha\tanh\beta)(\sinh\gamma b). \tag{3.11}$$

Next we define

$$F_-(\alpha, b) = \tfrac{1}{2}(\Phi_-(\alpha, b-0) - \Phi_-(\alpha, b+0) \tag{3.12}$$

and from equations (3.3), (3.10), (3.12), (3.5) and (3.7) we get

$$2F_-(\alpha, b) - \mathrm{i}\exp(-\mathrm{i}\,kb\sinh\beta)/(k\cosh\beta + \alpha) = [2A\gamma/(\gamma - \alpha\tanh\beta)]\,e^{\gamma b}. \tag{3.13}$$

Eliminating A from (3.13) and (3.11) we get

$$2F_-(\alpha, b) - \frac{\cosh^2\beta}{\alpha^2 - k^2\cosh^2\beta}\,\frac{b\gamma\,e^{\gamma b}}{b\sinh\gamma b}\,\Psi_+(\alpha, b) - \frac{\mathrm{i}\exp(-kb\sinh\beta)}{\alpha + k\cosh\beta} = 0, \tag{3.14}$$

which is a functional equation of the Wiener–Hopf type.

4. Solution of the Wiener–Hopf equation

We can write

$$(\sinh \gamma\beta)/\gamma b\ e^{\gamma b} = L(\alpha) = L_+(\alpha)\, L_-(\alpha) \tag{4.1}$$

where $L_+(\alpha)$ and $L_-(\alpha)$ are regular and non-zero in $\tau > -k_2$ and $\tau < k_2$ respectively, and both tend asymptotically to $|\alpha|^{-1/2}$ as $\alpha \to \infty$ in their appropriate half-planes of regularity. The factors $L_+(\alpha)$ and $L_-(\alpha)$ are known (Noble 1958) and their properties are not repeated here. We only mention that

$$L_+(-\alpha) = L_-(\alpha). \tag{4.2}$$

Equation (3.14) may now be rewritten as

$$2F_-(\alpha, b)(\alpha - k \cosh\beta)L_-(\alpha) - \frac{\cosh^2\beta}{b(\alpha + k\cosh\beta)L_+(\alpha)}\Psi_+(\alpha, b)$$
$$- \mathrm{i}\exp(-kb\sinh\beta)\frac{\alpha - k\cosh\beta}{\alpha + k\cosh\beta}L_-(\alpha) = 0. \tag{4.3}$$

Moreover we may write

$$\frac{\alpha - k\cosh\beta}{\alpha + k\cosh\beta}L_-(\alpha) = \frac{(\alpha - k\cosh\beta)L_-(\alpha) + 2k(\cosh\beta)L_-(-k\cosh\beta)}{\alpha + k\cosh\beta}$$
$$- \frac{2k(\cosh\beta)L_-(-k\cosh\beta)}{\alpha + k\cosh\beta}$$

where the first part is regular in $\tau < \tau_0$ and the second part is regular in $\tau > -\tau_0$. Equation (4.3) then becomes

$$2F_-(\alpha, b)\,(\alpha - k\cosh\beta)L_-(\alpha) - \frac{\mathrm{i}\exp(-kb\sinh\beta)}{\alpha + k\cosh\beta}$$
$$\times[(\alpha - k\cosh\beta)L_-(\alpha) + 2k(\cosh\beta)L_-(-k\cosh\beta)]$$
$$= \frac{\cosh^2\beta}{b(\alpha + k\cosh\beta)L_+(\alpha)}\Psi(\alpha, b) - \frac{2\mathrm{i}\exp(-kb\sinh\beta)}{\alpha + k\cosh\beta}L_-(-k\cosh\beta)\,k\cosh\beta. \tag{4.4}$$

In equation (4.4) the left-hand side is regular in $\tau < \tau_0$ while the right-hand side is regular in $\tau > -\tau_0$, and hence both sides are regular in the strip $|\tau| < \tau_0$.

Then by analytic continuation they define a function which is regular over the entire α plane. Using next the edge conditions (2.6), and considering the asymptotic behaviour of $\Psi_+(\alpha, b)$ and $F_-(\alpha, b)$ as $\alpha \to \infty$ in their appropriate half-planes of regularity, we find, on applying the appropriate Abel theorem, that each member of (4.4) must be zero, and hence

$$\Psi(\alpha, b) = [2\mathrm{i}kbL_+(k\cosh\beta)\exp(-kb\sinh\beta)/\cosh\beta]L_+(\alpha) \tag{4.5}$$

From relations (3.11), (3.5) and (3.7) we now obtain

$$A = \frac{2kbL_+(k\cosh\beta)\exp(-kb\sinh\beta)}{\cosh\beta}\,\frac{L_+(\alpha)}{(\gamma - \alpha\tanh\beta)(\sinh\gamma b)} \tag{4.6a}$$

$$B = \frac{2kbL_+(k\cosh\beta)\exp(-kb\sinh\beta)}{\cosh\beta}\,\frac{L_+(\alpha)}{(\gamma-\alpha\tanh\beta)(\sinh\gamma b)} \tag{4.6b}$$

$$C = \frac{-2\mathrm{i}kbL_+(k\cosh\beta)\exp(-kb\sinh\beta)}{\cosh\beta}\,\frac{L_+(\alpha)\,\mathrm{e}^{\gamma b}}{\gamma+\alpha\tanh\beta}. \tag{4.6c}$$

5. The solution $\phi(x, y)$

5.1. *Region A* $(0 \leq y \leq b, -\infty < x < \infty)$

From (3.3) and (4.6) we have

$$\Phi(\alpha, y) = E\frac{L_+(\alpha)}{(\sinh\gamma b)}\left(\frac{\mathrm{e}^{-\gamma y}}{\gamma+\alpha\tanh\beta}+\frac{\mathrm{e}^{\gamma y}}{\gamma-\alpha\tanh\beta}\right) \tag{5.1}$$

where

$$E = \mathrm{i}kL_+(k\cosh\beta)\exp(-kb\sinh\beta)/\cosh\beta. \tag{5.2}$$

The solution $\phi(x, y)$ for this region is expressed by the Fourier inverse

$$\phi(x, y) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}a}^{\infty+\mathrm{i}a}\Phi(\alpha, y)\,\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha \qquad (|a| < \tau_0). \tag{5.3}$$

The contour of integration is shown in figure 2.

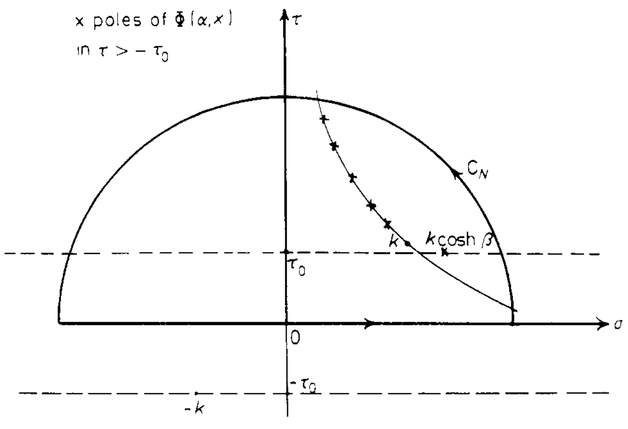

Figure 2. The strip of regularity of $\Phi(\alpha, y)$ and the contour of integration for the integral in equation (5.3).

For $x < 0$ we close the contour in the upper half-plane where the only singularities of $\Phi(\alpha, y)$ are poles located at $\alpha_k = k\cosh\beta$ and $\alpha_n = \mathrm{i}(n^2\pi^2/b^2 - k^2)^{1/2}$, $n = 1, 2, \ldots$. It can be shown that if the contour is closed by a sequence of concentric circular arcs C_N, $N = 1, 2, \ldots$, such that each arc C_N, whose equation is $|\alpha| = |R_N|$, passes through no pole of $\Phi(\alpha, y)$, and the radius R_N tends to infinity as $N \to \infty$, then $\Phi(\alpha, y) \to 0$ on C_N as $N \to \infty$. Hence $\int_C \Phi(\alpha, y)\,\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha = 0$ on C, where C is the above limit of C_N, $N \to \infty$, and thus an application of the residue theorem gives

$$\phi(x, y) = \mathrm{i}\sum \mathrm{Res}(\Phi(\alpha, y)\,\mathrm{e}^{-\mathrm{i}\alpha x}). \tag{5.4}$$

By calculating the residues at the above poles we get

$$\phi(x, y) = \frac{-kb(\sinh\beta)\mathrm{L}_+(k\cosh\beta)\exp(-kb\sinh\beta)}{\sinh(kb\sinh\beta)}\exp[k(-\mathrm{i}x\cosh\beta + y\sinh\beta)]$$
$$+\frac{\mathrm{i}E\pi}{b^2}\sum_{n=1}^{\infty}\frac{n(-1)^n}{\alpha_n}L_+(\alpha_n)$$
$$\times\left(\frac{\mathrm{e}^{(\mathrm{i}\pi n/b)y}}{(\pi n/b)+\mathrm{i}\alpha_n\tanh\beta}+\frac{\mathrm{e}^{-(\mathrm{i}\pi n/b)y}}{(\pi n/b)-\mathrm{i}\alpha_n\tanh\beta}\right)\mathrm{e}^{-\mathrm{i}\alpha_n x}. \tag{5.5}$$

The first term in equation (5.5), coming from the pole $\alpha_k = k\cosh\beta$, is the 'reflected' Kelvin wave moving to the left inside the semi-infinite channel. The amplitude of this wave, as a function of k, β and b, depends on the depth (h) and the breadth (b) of the channel as well as on the frequency (ω) of the incident wave and the latitude (ρ) of the water sheet considered.

The remaining part of the solution in (5.5) is a convergent series containing all the possible modes propagating along the channel in the negative x direction.

For $bk < \pi$ all α_n's are pure imaginary and positive, and hence all the modes represent attenuated waves, the amplitude of which decays exponentially to zero as $x \to -\infty$.

Let it be noted finally that in the event that $k = \pi n/b$ we have $\alpha_n = 0$, and hence the amplitude of the mode of the nth term of the series in (5.5) becomes infinite (resonance). Thus we must suppose that k is sufficiently far from $\pi n/b$, $n = 1, 2, \ldots$. For $f = 0$ the solution corresponds to the usual modes without rotation and also represents the propagation of two-dimensional sound waves or electromagnetic waves of TE type.

For $x > 0$ the solution may be determined from the solution for region B given in § 5.2, which could be continuously extended to this part of region A.

5.2. *Region* B ($y \geqslant b, -\infty < x < \infty$)

Again from (3.3) and (4.6) we get

$$\Phi(\alpha, y) = -2E\,[L_+(\alpha)/(\gamma + \alpha\tanh\beta)]\,\mathrm{e}^{-\gamma(y-b)} \tag{5.6}$$

and

$$\phi(x, y) = -\frac{E}{\pi}\int_{-\infty+\mathrm{i}a}^{\infty+\mathrm{i}a}\frac{L_+(\alpha)}{\gamma+\alpha\tanh\beta}\,\mathrm{e}^{-\gamma(y-b)-\mathrm{i}\alpha x}\,\mathrm{d}\alpha \qquad (|a| < \tau_0). \tag{5.7}$$

The contour Γ of integration is shown in figure 3. We take $k_2 \to 0+0$ for the remainder of this paper to avoid unnecessary complications.

The integrand in (5.7) has a pole at $\alpha = -k\cosh\beta$. It has also branch points at $\alpha = \pm k$, and so the above integral cannot be evaluated in a closed form.

However, we can determine the leading term of the asymptotic expansion of the integral in (5.7) for the far field by applying the method of steepest descents (Copson 1970).

The transformations

$$x = r\cos\theta, \qquad y - b = r\sin\theta \qquad (r > 0,\ 0 < \theta < \pi) \tag{5.8a}$$

and

$$\alpha = -k\cos z, \qquad \gamma = -ik\sin z, \qquad z = p + \mathrm{i}q \qquad (p, q \text{ real}) \tag{5.8b}$$

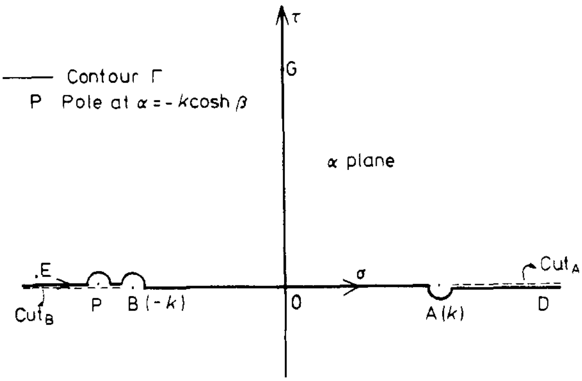

Figure 3. Contour Γ for the integral in equation (5.7) as $k_2 \to 0+0$

are applied to the integral in (5.7), and the result is

$$\phi(x, y) = \frac{E}{\pi} \int_{\Gamma'} \frac{L_+(-k \cos z) \sin z}{\mathrm{i} \sin z + (\cos z) \tanh \beta} \mathrm{e}^{\mathrm{i}kr \cos(z-\theta)} \, \mathrm{d}z. \tag{5.9}$$

The new contour Γ' of integration in the z plane is shown in figure 4.

By the transformation (5.9) the whole α plane is mapped into the strip $(0 \leqslant p \leqslant \pi, -\infty < q < +\infty)$ of the z plane. In figure 4 the points 0′, A′, B′, D′, E′ are images of the points 0, A, B, D, E respectively of figure 3.

The relation $\mathrm{d}W/\mathrm{d}z = 0$, where $W = \mathrm{i}\cos(z-\theta)$, determines the possible saddle points, which are $z = \theta + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, and we choose the saddle point S at $z = \theta$ since it lies inside the above strip. The possible paths of steepest descents are involved in the equation $\mathrm{Im}\{W\} = \mathrm{const} = \mathrm{Im}\{W\}_{z=0}$ which gives

$$\cos(p-\theta) \cosh q = 1. \tag{5.10}$$

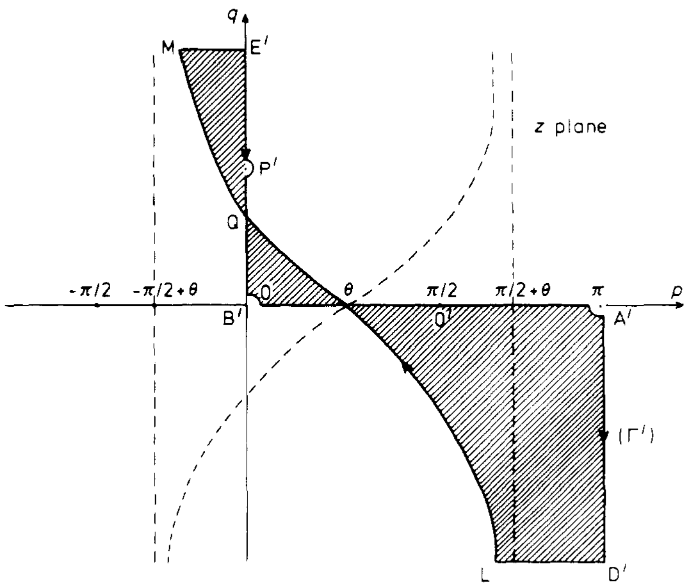

Figure 4. Contour Γ′ and the steepest descent path for the integral in equation (5.9).

Of the two branches which are involved in (5.10), that which crosses the positive p axis at an angle of $\frac{3}{4}\pi$ and on which $\mathrm{Re}\{W\}$ has a maximum at S is the path of steepest descent (LSM) as shown in figure 4. The asymptotes of this path are the lines $z = \theta \pm \frac{1}{2}\pi$. The oscillation of the integrand in (5.9) varies slowly on LSM, as the phase of e^{krW} remains stationary. On the other hand, the integrand, having a maximum at S, decreases rapidly on each side of S on LSM. So the main contribution to the integral is due to a small segment of LSM around the saddle point where the integrand can be taken to be constant. Let us now consider the closed contour B′A′D′LSME′B′; the integrand in (5.9) has a pole P′ at $z = \mathrm{i}\beta$ (figure 4), and it can easily be seen that this pole is inside the above closed contour if $0 < \theta < \cos^{-1}(kc/\omega) < \frac{1}{2}\pi$.

Applying now the Cauchy theorem we get for the above closed contour

$$\int_{\mathrm{MB}'} + \int_{\Gamma'} + \int_{\mathrm{D'L}} + \int_{\mathrm{LSM}} \frac{E}{\pi}\, \frac{L_+(-k\cos z)\sin z}{\mathrm{i}\sin z + (\cos z)\tanh\beta}\, e^{\mathrm{i}kr\cos(z-\theta)}\, \mathrm{d}z = -2\mathrm{i}\pi a_{-1} H(\theta_0 - \theta) \tag{5.11}$$

where a_{-1} is the residue of the pole P' and is found to be

$$(E/2\pi)(\sinh 2\beta) L_+(-k\cosh\beta)\, e^{k(\mathrm{i}z\cosh\beta - y\sinh\beta)};$$

H is the Heaviside unit function; and $\theta_0 = \cos^{-1}(kc/\omega)$ $(0 < \theta < \frac{1}{2}\pi)$. On the other hand, it can be seen that the contribution of the segments D′L and ME′ into (5.11) is zero as $|q| \to \infty$.

Moreover the integral on LSM may be now rewritten as

$$\frac{E}{\pi}\, \frac{L_+(\cos\theta)\sin\theta}{\mathrm{i}\sin\theta + (\cos\theta)\tanh\beta} \int_{\mathrm{LSM}} e^{\mathrm{i}kr\cos(z-\theta)}\, \mathrm{d}z.$$

Yet the last integral equals $-\pi H_0^{(1)}(kr)$, where $H_0^{(1)}(kr)$ is a Hankel function of the first kind and of order zero, as can easily be recognised if we go back to the contour Γ', and, since the asymptotic expression for large kr is

$$H_0^{(1)}(kr) \sim (2/\pi kr)^{1/2}\, e^{\mathrm{i}(kr-\pi/4)},$$

equation (5.11) gives

$$\phi(x, y) \sim E \frac{L_+(-k\cos\theta)}{\mathrm{i} + (\cot\theta)\tanh\beta} \left(\frac{2}{\pi kr}\right)^{1/2} e^{\mathrm{i}(kr-\pi/4)} - H(\theta - \theta_0)\mathrm{i}E(\sinh 2\beta) L_+(-k\cosh\beta)\, e^{k(\mathrm{i}x\cosh\beta - y\sinh\beta)}. \tag{5.12}$$

The first term of the solution (5.12) is a cylindrical wave (Poincaré wave) coming from the edge $(0, b)$, and the second term is a Kelvin wave travelling to the right in $x > 0$. Since, for larger r, the cylindrical wave is of order $(r^{-1/2})$, while the Kelvin wave is of order $e^{-(f/c)y}$, we conclude that when y is small (the region near the barriers) the leading term in (5.12) is the Kelvin wave, as it does not diminish with distance in the x direction; when, however, y also becomes large, the Poincaré wave becomes the dominant term expressing asymptotically the surface elevation considered.

Let it be noted finally that for $f = 0$ (no rotation) the Kelvin wave disappears, and the rest of the solution represents the solution of the corresponding problems in acoustics and electromagnetism.

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References

Copson E T 1970 *Asymptotic Expansions* (Cambridge: University Press)
Heins A E 1956 *Comm. Pure Appl. Math.* **9** 447–66
Kapoulitsas G M 1975 *PhD Thesis* University of Surrey
Mittra R and Lee S W 1971 *Analytical Techniques in the Theory of Guided Waves* (London: Macmillan)
Noble B 1958 *Methods Based on the Wiener–Hopf Technique* (Oxford: Pergamon)
Proudman J 1953 *Dynamical Oceanography* (London: Methuen)